Fuzzy logic in intelligent adaptable systems

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Probability Function based on Flows and Tensions

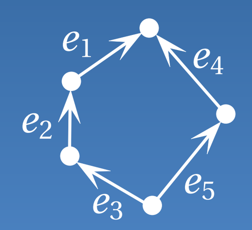
modular tension polynomial
$\bar{\theta}_G(k) = \#\text{nowhere-zero } \mathbb{Z}_k\text{-tensions}$

integral tension polynomial
$\theta_G(k) = \#\text{nowhere-zero } k\text{-tensions}$

modular flow polynomial
$\bar{\varphi}_G(k) = \#\text{nowhere-zero } \mathbb{Z}_k\text{-flows}$

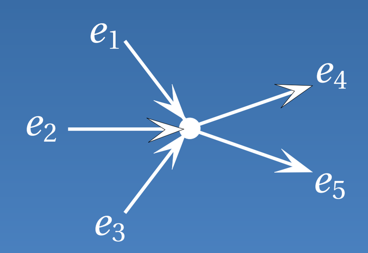
integral flow polynomial
$\varphi_G(k) = \#\text{nowhere-zero } k\text{-flows}$

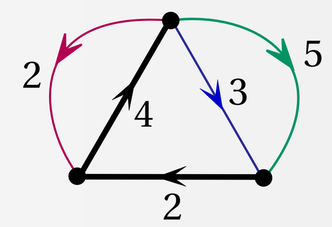
$\mathbb{Z}_k$-**tension** $t : E \to \mathbb{Z}_k$
k-**tension** $t : E \to \{-k+1, \ldots, k-1\}$
such that along every cycle tension is conserved



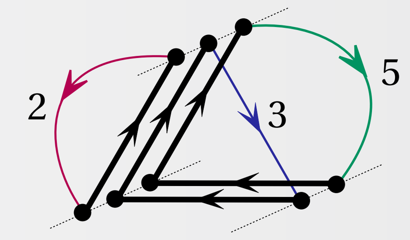
$f(e_1) + f(e_2) + f(e_3) = f(e_4) + f(e_5)$

$\mathbb{Z}_k$-**flow** $f : E \to \mathbb{Z}_k$
k-**flow** $f : E \to \{-k+1, \ldots, k-1\}$
such that at every vertex flow is conserved

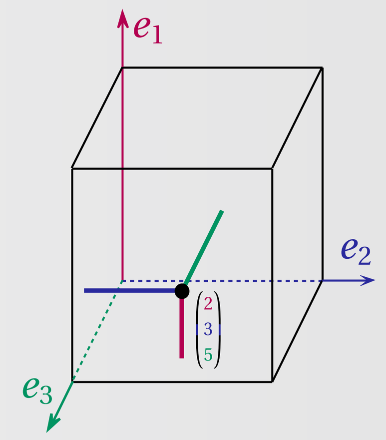




a $\mathbb{Z}_6$-flow on G



a flow is determined by its values on the non-tree edges



view $\mathbb{Z}_6$-flow as a point $f \in \mathbb{Z}^{E\setminus T} \cap 6 \cdot [0,1)^{E\setminus T}$

flow on non-tree edges

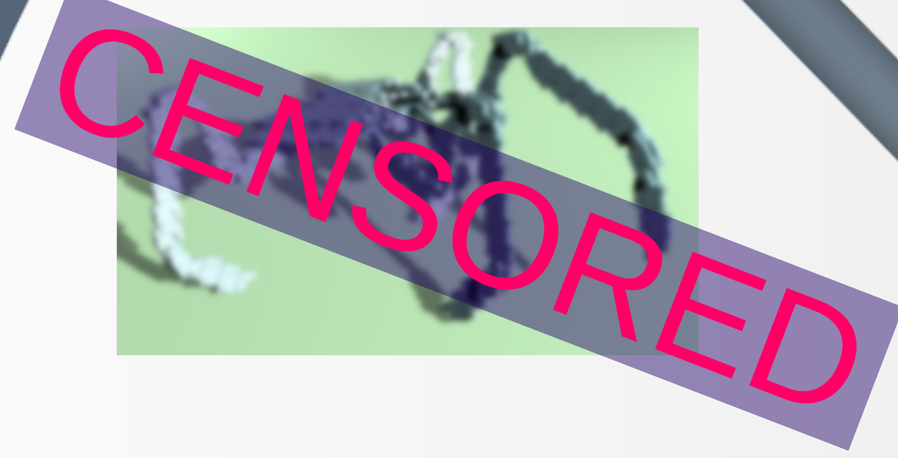
$$\begin{matrix} & e_1\ e_2\ e_3 \\ \begin{matrix} e_4 \\ e_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{matrix} f|_{E\setminus T} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = f|_T$$

network matrix — flow on tree edges

Motivation

Our system is used for military purposes.
The system is based on intelligent adaptable robots that are able to scan surrounding area and adapt for several conditions.
What was missing in past was support for fuzzy data that can be obtained as a result from several algorithms used by robot's geographical information system.

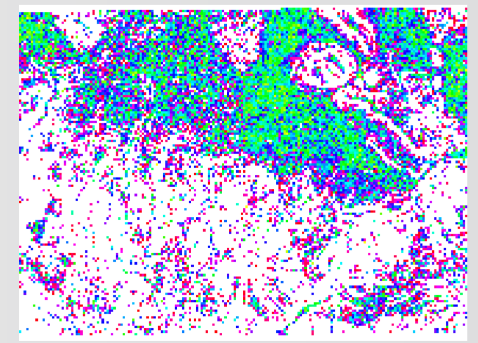
Robot



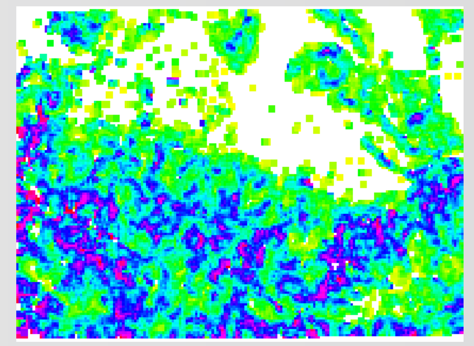
Example

It is based on free Spearfish database
It is based on new fuzzy modules in GRASS GIS developed for purposes of our project
Modules are: r.slope.fuzzy, r.landcover.fuzzy, r.cost.fuzzy

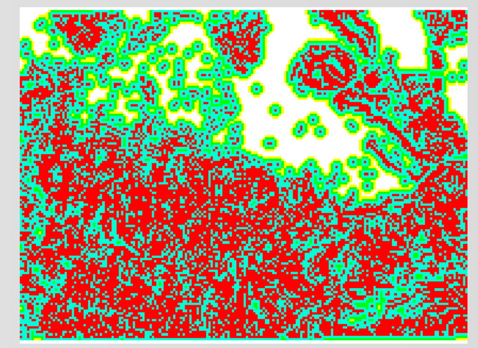
Slope 0-10 deegrees



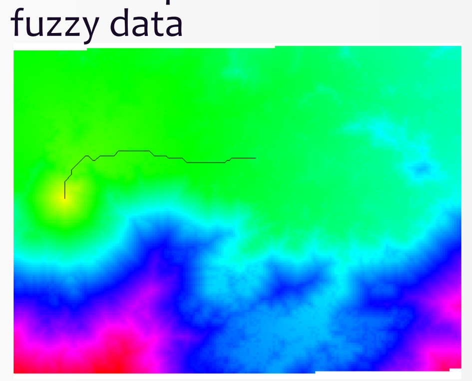
Slope 0-10 deegrees with fuzzy distribution



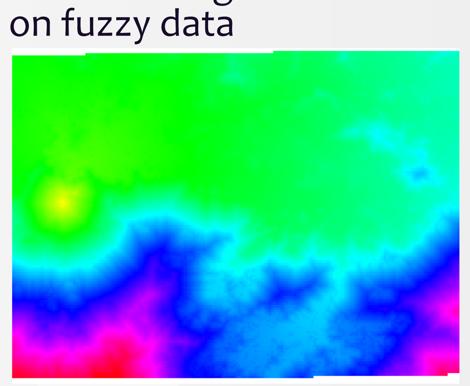
Probablity function based on flows and tensions



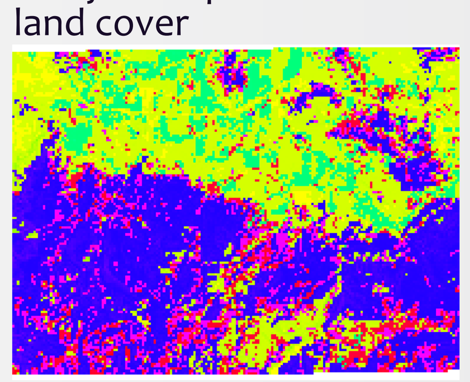
Shortest path based on fuzzy data



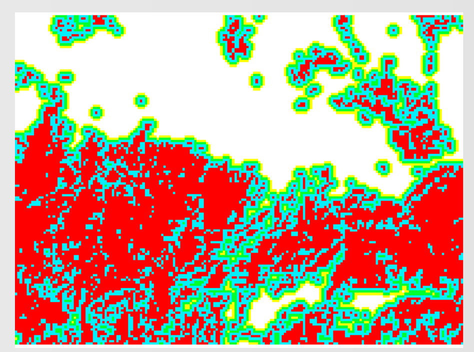
Basic cost algortihtm runned on fuzzy data



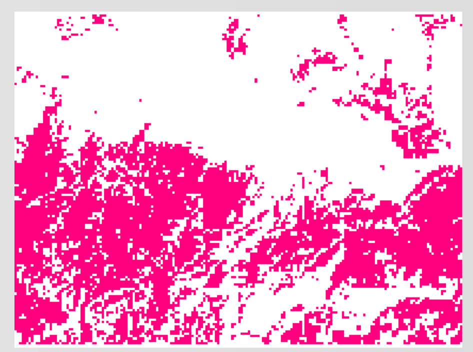
Friction surface based on fuzzy of slope and land cover



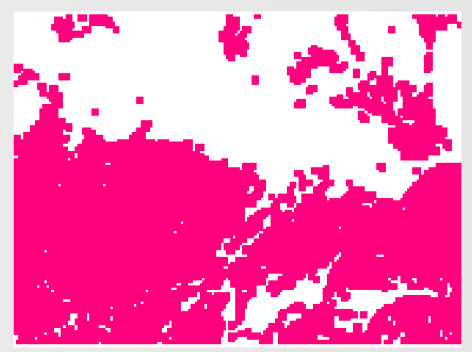
Probability function based on flows and tensions



Deciduous forests



Deciduous forests with fuzzy distribution



Relative Polytopal Complexes

A d-dimensional polytope P is **integral** if all pixels of P have integer coordinates.
P is called **compressed** if every pulling interpolation of P is unimodular.
A **relative polytopal complex** is a pair $C' \subseteq C$ of polytopal complexes.

$\bigcup C \setminus \bigcup C'$ is the set of points $x \in \mathbb{R}^n$ contained in C but not in C', that is,

$$\bigcup C \setminus \bigcup C' = \bigcup_{P \in C} P \setminus \bigcup_{P' \in C'} P'.$$

For any set $X \subset \mathbb{R}^n$ the **Ehrhart function** $\mathrm{L}_X : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0}$ is given by
$\mathrm{L}_X(k) := \#\mathbb{Z}^n \cap k \cdot X.$

Fact. If $C' \subseteq C$ is an integral relative polytopal complex, then $\mathrm{L}_{\bigcup C \setminus \bigcup C'}(k)$ is a polynomial in k called the **Ehrhart polynomial** of $C' \subseteq C$.

Relative Stanley-Reisner Ideals

Stanley-Reisner ideal
$I_\Delta := \langle x^u \mid \mathrm{supp}(u) \notin \Delta \rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]$

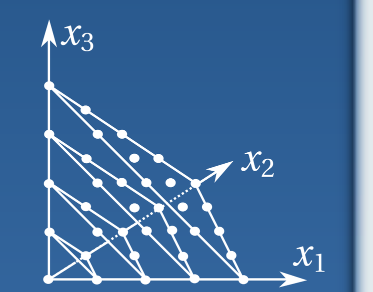
Stanley-Reisner ring
$\mathbb{K}[\Delta] := \mathbb{K}[x_1, \ldots, x_n] / I_\Delta$

Relative Stanley-Reisner ideal
$I_{\Delta/\Delta'} := \langle x^u \mid \mathrm{supp}(u) \notin \Delta' \rangle \subseteq \mathbb{K}[\Delta]$

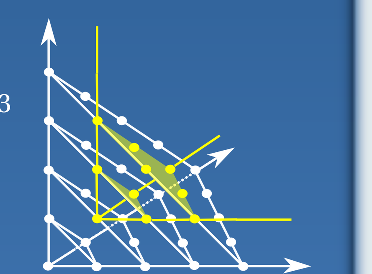
The **Hilbert function**
$\mathrm{H}_{I_{\Delta/\Delta'}}(k)$ counts monomials of degree k in $I_{\Delta/\Delta'} \subseteq \mathbb{K}[\Delta]$.

Example:

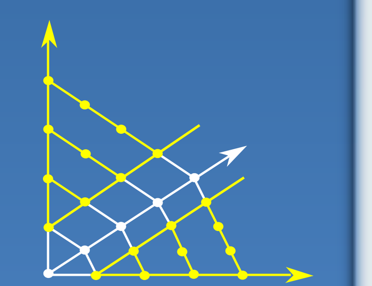
$\mathbb{K}[x] := \mathbb{K}[x_1, x_2, x_3]$



$\Delta = $ x_1 — x_2 — x_3
$I_\Delta = \langle x_1 x_3 \rangle$



$\Delta' = $ x_2
$I_{\Delta/\Delta'} = \langle x_1, x_3 \rangle \subset \mathbb{K}[x] / I_\Delta$



The Hilbert function of $I_{\Delta/\Delta'}$ is $H_{I_{\Delta/\Delta'}}(k) = 2k.$

Bounds on the Coefficients

Theorem.
A polynomial $f(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i}$ is the Hilbert function of some relative Stanley-Reisner ideal *if and only if*

$f_i \in \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq d.$

Better Bounds on the Coefficients

...exploiting the geometry of inside-out polytopes.

Theorem.
Let p denote the modular flow or tension polynomial of a graph.
Let $d+1 = \deg p$ and define the h-vector $(h_0, \ldots, h_{d+1})$ of the polynomial $(k+1)^{d+1} - p(k)$ by

$$1 + \sum_{k \geq 1} \left((k+1)^{d+1} - p(k)\right) z^k = \frac{h_0 z^0 + \ldots h_{d+1} z^{d+1}}{(1-z)^{d+1}}.$$

Then
1. $h_0 \leq h_1 \leq \ldots \leq h_{\lfloor d/2 \rfloor}$,
2. $h_i \leq h_{d-i}$ for $i \leq d/2$,
3. $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-vector.

Hilbert vs. Ehrhart - Tension in fuzzy

Theorem.
Let C be a polytopal complex.
If all faces of C are compressed lattice polytopes, then
for any subcomplex $C' \subset C$ there exists a relative Stanley-Reisner ideal $I_{\Delta,\Delta'}$ such that for all $k \in \mathbb{Z}_{\geq 0}$

$$\mathrm{L}_{\bigcup C \setminus \bigcup C'}(k) = \mathrm{H}_{I_{\Delta/\Delta'}}(k).$$

Ehrhart polynomial — Hilbert function